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Entropies of Static Dilaton Black Holes From the Cardy Formula

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A standard Virasoro subalgebra for a static dilaton black hole obtained in the low-energy effective field theory describing heterotic string is constructed at a Killing horizon. The statistical entropies of the Garfinkle–Horowitz–Strominger dilaton black hole and the Gibbons–Maeda dilaton black hole obtained by standard Cardy formula agree with their Bekenstein–Hawking entropies only if we take period T of function υ as the periodicity of the Euclidean black hole. We also consider first-order quantum correction to the entropy and find that the correction is described by a logarithmic term with a factor of $-\frac{1}{2}$, which is different from Kaul and Majumdar's factor of $-\frac{3}{2}$.

1. INTRODUCTION

The statistical mechanical description of the Bekenstein–Hawking black hole entropy (Bekenstein, 1972, 1973, 1974; Hawking, 1974, 1975; Kallosh *et al.*, 1993) both in string theory (Youm, 1999) and in "quantum geometry" (Ashtekar *et al.*, 1998) has attracted much attention recently. Carlip (1999a,b) derived the central extension of the constraint algebra of general relativity by Brown–Henneaux's approach and manifested covariant phase space methods (Iyer and Wald, 1994, 1995; Lee and Wald, 1990; Wald, 1993), and found that a natural set of boundary conditions on the (local) Killing horizon leads to a Virasoro subalgebra with a calculable central charge and the standard Cardy formula gives the Bekenstein–Hawking entropies. Those works suggested that the asymptotic behavior of the density of states may be determined by the algebra of diffeomorphism at horizon. Recently, we (Jing and Yan, in press) extended Carlip's investigation (Carlip, 1999b) for vacuum case to a case including a cosmological term and electromagnetic fields and calculated the statistical entropies of Kerr–Newman black hole and Kerr–Newman–AdS black hole using standard Cardy formula.

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Kaul and Majumdar (2000) found that the lowest order corrections to the Bekenstein–Hawking entropy is shown by a logarithmic term

$$S \sim \frac{A_{\rm H}}{4} - \frac{3}{2}\ln\frac{A_{\rm H}}{4} + \text{const} + \cdots, \tag{1.1}$$

where $A_{\rm H}$ is the area of the black hole. Carlip (2000) also studied the quantum corrections to the Cardy formula and found that the entropy is given by

$$S \sim S_0 - \frac{3}{2}\ln S_0 + \ln c + \text{const} + \cdots, \tag{1.2}$$

where S_0 is standard Bekenstein–Hawking entropy and c is a central charge of a Virasoro subalgebra. Carlip suggested that if the central charge is used in the sense of being independent of the horizon area (Carlip thinks that this can be done by adjusting the periodicity β (Carlip, 2000)), then the factor of $-\frac{3}{2}$ in logarithmic term would be universal.

In this paper we will investigate whether or not the Carlip's conclusion (the asymptotic behavior of the density of states may be determined by the algebra of diffeomorphism at horizon) and Kaul and Majumdar's result (the leading corrections to the entropy is a logarithm of the horizon area with a factor of $-\frac{3}{2}$) are valid for the static dilation black holes. In Section 2, using the covariant phase techniques a constraint algebra is constructed for the gravity coupled to a Maxwell field and a dilaton. In Section 3, standard Virasoro subalgebras are obtained for Garfinkle–Horowitz–Strominger dilaton black hole and Gibbons–Maeda dilaton black hole and then statistical entropies are calculated using the standard Cardy formula and the quantum corrections to the entropy by the Cardy formula, respectively. The last section devotes to summary.

2. CONSTRAINT ALGEBRA ON THE KILLING HORIZON

Lee *et al.* (Iyer and Wald, 1994, 1995; Lee and Wald, 1990; Wald, 1993) showed that for an infinitesimal generator ξ^a of a diffeomorphism the Lagrangian $\mathbf{L}$, equation of motion n-form $\mathbf{E}$, symplectic potential $(n-1)$-form $\boldsymbol{\Theta}$, Noether current $(n-1)$-form $\mathbf{J}$, and Noether charge $(n-2)$-form $\mathbf{Q}$ satisfy

$$\delta\mathbf{L} = \mathbf{E}\delta\phi + d\boldsymbol{\Theta}, \tag{2.1}$$

$$\mathbf{J}[\xi] = \boldsymbol{\Theta}[\phi, \mathcal{L}_\xi\phi] - \xi\cdot\mathbf{L}, \tag{2.2}$$

$$\mathbf{J} = d\mathbf{Q}. \tag{2.3}$$

Hamilton's equation of motion is given by

$$\delta H[\xi] = \int_C \omega[\phi, \delta\phi, \mathcal{L}_\xi\phi] = \int_C [\delta\mathbf{J}[\xi] - d(\xi\cdot\boldsymbol{\Theta}[\phi, \delta\phi])]. \tag{2.4}$$

Using Eq. (2.3) and defining a $(n-1)$-form $\mathbf{B}$ as $\delta \int_{\partial C} \xi \cdot \mathbf{B}[\phi] = \int_{\partial C} \xi \cdot \mathbf{\Theta}[\phi \cdot \delta\phi]$, the Hamiltonian can then be expressed (Carlip, 1999b)

$$H[\xi] = \int_{\partial C} (\mathbf{Q}[\xi] - \xi \cdot \mathbf{B}[\phi]). \tag{2.5}$$

The Poisson bracket forms a standard "surface deformation algebra" (Brown and Henneaux, 1986; Carlip, 1999b)

$$\{H[\xi_1], H[\xi_2]\} = H[\{\xi_1, \xi_2\}] + K[\xi_1, \xi_2], \tag{2.6}$$

where the central term $K[\xi_1, \xi_2]$ depends on the dynamical fields only through their boundary values.

The low-energy Lagrangian obtained from heterotic string theory in four dimensional spacetime is described by

$$\mathbf{L}_{abcd} = \epsilon_{abcd}[R - 2(\nabla\phi)^2 - e^{-2\alpha\phi}F^2], \tag{2.7}$$

where ϵ_{abcd} is the volume element, ϕ is the dilaton scalar field, F_{ab} is the Maxwell field associated with a $U(1)$ sub-group of $E_8 \times E_8$ or Spin $(32)/Z_2$, and α is a free parameter which governs the strength of the coupling of the dilaton to the Maxwell field. From Lagrangian (2.7) we find that the equations of motion $\mathbf{E}$ for dynamical fields A_μ, ϕ, and $g_{\mu\nu}$ can be respectively expressed as

$$\nabla_\mu \left(e^{-2\alpha\phi}F^{\mu\nu}\right) = 0, \tag{2.8}$$

$$\nabla^2\phi + \frac{1}{2}\, e^{-2\alpha\phi}F_{\mu\nu}F^{\mu\nu} = 0, \tag{2.9}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}(\nabla\phi)^2 + 2\, e^{-2\alpha\phi}F_{\beta\nu}F^{\beta}_{\mu} - \frac{1}{2}g_{\mu\nu}\, e^{-2\alpha\phi}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.10}$$

We know from Eq. (2.2) that the symplectic potential $(n-1)$-form is given by

$$\mathbf{\Theta}_{bcd}[g, \mathcal{L}_\xi g] = 4\epsilon_{abcd}\left\{\frac{1}{2}\left(\nabla_e\nabla^{[e}\xi^{a]} + R^a_e\xi^e\right) - \xi^e\nabla_e\phi\nabla^a\phi - e^{-2\alpha\phi}F^{af}[F_{ef}\xi^e + (\xi^e A_e)_{;f}]\right\}. \tag{2.11}$$

Eqs. (2.2) and (2.11) show

$$\mathbf{J}_{bcd} = 2\epsilon_{abcd}\left\{\nabla_e\nabla^{[e}\xi^{a]} - 2\, e^{-2\alpha\phi}F^{af}(\xi^e A_e)_{;f} + \left[R^a_e - \frac{1}{2}\delta^a_e R - 2\nabla_e\phi\nabla^a\phi + \delta^a_e(\nabla\phi)^2 - 2\, e^{-2\alpha\phi}F^{af}F_{ef} + \frac{1}{2}\delta^a_e\, e^{-2\alpha\phi}F^2\right]\xi^e\right\}$$

$$= 2\epsilon_{abcd}\big[\nabla_e \nabla^{[e}\xi^{a]} - 2\, e^{-2\alpha\phi} F^{af} (\xi^e A_e)_{;f}\big]$$
$$= 2\epsilon_{abcd}\big[\nabla_e \nabla^{[e}\xi^{a]} + 4\nabla_f \big(e^{-2\alpha\phi} \nabla^{[f} A^{a]} A_e \xi^e\big)\big]. \tag{2.12}$$

We used the equations of motion (2.10) and (2.8) in above calculation. Using Eqs. (2.3) and (2.12) we obtain

$$\mathbf{Q}_{cd} = -\epsilon_{abcd}[\nabla^a \xi^b + 4\, e^{-2\alpha\phi} A_e \xi^e \nabla^a A^b]. \tag{2.13}$$

For a static dilaton black hole, the dilation scalar field, the electromagnetic potential A_a, and the Killing vector can be respectively expressed as

$$\phi = \phi(r), \tag{2.14}$$

$$A_a = (A_0(\tau, \theta), 0, 0, A_3(r, \theta)), \tag{2.15}$$

$$\chi^a_{\mathrm{H}} = (1, 0, 0, 0). \tag{2.16}$$

Similar to Carlip's definition (Carlip, 1999b), we define a "stretched horizon" $\chi^2 = \epsilon$, where $\chi^2 = g_{ab}\chi^a\chi^b$, χ^a is a Killing vector. The result of the computation will be evaluated at the event horizon of the black hole by taking ϵ to zero. Near the stretched horizon, one can introduce a vector orthogonal to the orbit of χ^a by $\nabla_a \chi^2 = -2\kappa\rho_a$ where κ is the surface gravity. The vector ρ^a satisfies conditions

$$\chi^a \rho_a = -\frac{1}{\kappa}\chi^a \chi^b \nabla_a \chi_b = 0, \quad \text{everywhere}$$
$$\rho^a \to \chi^a, \quad \text{at the horizon.} \tag{2.17}$$

To preserve "asymptotic" structure at horizon, we impose Carlip's boundary conditions (Carlip, 1999b)

$$\delta\chi^2 = 0, \quad \chi^a t^b \delta g_{ab} = 0, \quad \delta\rho_a = -\frac{1}{2\kappa}\nabla_a(\delta\chi^2) = 0, \qquad \text{at} \quad \chi^2 = 0, \tag{2.18}$$

where t^a is a any unit spacelike vector tangent to boundary $\partial\mathbf{M}$ of the spacetime $\mathbf{M}$. And the infinitesimal generator of a diffeomorphism is taken as

$$\xi^a = \mathcal{R}\rho^a + \mathcal{T}\chi^a, \tag{2.19}$$

where functions $\mathcal{R}$ and $\mathcal{T}$ obey the relations (Carlip, 1999b)

$$\mathcal{R} = \frac{1}{\kappa}\frac{\chi^2}{\rho^2}\chi^a \nabla_a \mathcal{T}, \quad \text{everywhere}$$
$$\rho^a \nabla_a \mathcal{T} = 0, \quad \text{at the horizon.} \tag{2.20}$$

For a one-parameter group of diffeomorphism such that $D\mathcal{T}_\alpha = \lambda_\alpha \mathcal{T}_\alpha$ ($D \equiv \chi^a \partial_a$), one introduces an orthogonality relation (Carlip, 1999b)

$$\int_{\partial C} \hat{\epsilon}\, \mathcal{T}_\alpha \mathcal{T}_\beta \sim \delta_{\alpha+\beta}, \tag{2.21}$$

where ∂C represents Killing horizon. The technical role of the condition (2.21) is to guarantee the existence of generators $H[\xi]$. By using the other future-directed null normal vector $N^a = k^a - \alpha\chi^a - t^a$, with $k^a = -\frac{1}{\chi^2}(\chi^a - \frac{|\chi|}{\rho}\rho^a)$ and a normalization $N_a\chi^a = -1$, the volume element can be expressed as

$$\epsilon_{abcd} = \hat{\epsilon}_{cd}(\chi_a N_b - \chi_b N_a) + \cdots, \tag{2.22}$$

the omitted terms do not contribute to the integral.

We know that the first two terms in the right hand for the following equation

$$\int_{\partial C} \xi^b \boldsymbol{\Theta}_{bcd} = 4\int_{\partial C} \epsilon_{abcd}\xi^a \left\{ \frac{1}{2}\left(\nabla_e \nabla^{[e}\xi^{b]} + R^b_e \xi^e\right) - \xi^e \nabla_e\phi\nabla^b\phi \right.$$
$$\left. - e^{-2\alpha\phi} F^{fb}[F_{ef}\xi^e + (\xi^e A_e)_{;f}] \right\}, \tag{2.23}$$

can be treated as Carlip did (Carlip, 1999b). At the horizon, using Eqs. (2.14), (2.15), (2.16), and (2.18)–(2.22) we can show

$$\epsilon_{abcd}\xi_2^a\xi_1^e\nabla^b\phi\nabla_e\phi = 0, \tag{2.24}$$

$$\begin{aligned} &\epsilon_{abcd}\, e^{-2\alpha\phi}\xi^a F^{bf}[F_{ef}\xi^e + (\xi^e A_e)_{;f}] \\ &= \epsilon_{abcd}\, e^{-2\alpha\phi}\xi^a F^{bf}\delta_\xi A_f \\ &= \hat{\epsilon}_{cd}\, e^{-2\alpha\phi}\left[\frac{|\chi|}{\rho}\mathcal{T}\rho_b + \left(\frac{\rho}{|\chi|} + t\cdot\rho\right)\mathcal{R}\chi_b\right] F^{bf}\delta_\xi A_f \\ &= 0. \end{aligned} \tag{2.25}$$

Thus, the last three terms in Eq. (2.23) give no contribution to the central term.

By the boundary conditions we can prove

$$\epsilon_{abcd}\, e^{-2\alpha\phi} A_e\xi^e\nabla^a A^b \to 0. \tag{2.26}$$

Then, Eq. (2.13) becomes

$$Q_{cd} = -\epsilon_{abcd}\nabla^a\xi^b. \tag{2.27}$$

For the Noether current we have $\delta_{\xi_2}\mathbf{J}[\xi_1] = d[\xi_2(\boldsymbol{\Theta}[\phi, \mathcal{L}_{\xi 1}\phi] - \xi_1 \cdot \mathbf{L})]$. where δ_ξ denotes the variation corresponding to diffeomorphism generated by ξ. Substituting it into Eq. (2.4) and using Eq. (2.11) we obtain

$$\delta_{\xi_2} H[\xi_1] = \int_{\partial C} (\xi_2\boldsymbol{\Theta}[\phi, \mathcal{L}_{\xi_1}\phi] - \xi_1\boldsymbol{\Theta}[\phi, \mathcal{L}_{\xi_2}\phi] - \xi_2\xi_1\mathbf{L})$$
$$= \int_{\partial C} \epsilon_{abcd}\left[\xi_2^a\nabla_e\left(\nabla^e\xi_1^b - \nabla^b\xi_1^e\right) - \xi_1^a\nabla_e\left(\nabla^e\xi_2^b - \nabla^b\xi_2^e\right)\right]$$
$$- 4\int_{\partial C} \epsilon_{abcd}\, e^{-2\alpha\phi}\{\xi_2^a F^{fb}\left[F_{ef}\xi_1^e + \left(\xi_1^e A_e\right)_{;f}\right]$$

$$
\begin{aligned}
&- \xi_1^a F^{fb}\left[F_{ef}\xi_2^e + \left(\xi_2^e A_e\right)_{;f}\right]\Big\} \\
&- \int_{\partial C} \epsilon_{abcd} \left[4R_e^b\left(\xi_1^a\xi_2^e - \xi_2^a\xi_1^e\right) + \xi_2^a\xi_1^b\mathbf{L}\right] \\
&- 4\int_{\partial C} \epsilon_{abcd} \left(\xi_2^a\xi_1^e - \xi_1^a\xi_2^e\right)\nabla^b\phi\nabla_e\phi. \qquad (2.28)
\end{aligned}
$$

Using Eqs. (2.14)–(2.16) and (2.18)–(2.22) we know

$$
\begin{aligned}
&\int_{\partial C} \epsilon_{abcd}\left(\xi_2^a\xi_1^e - \xi_1^a\xi_2^e\right)\nabla^b\phi\nabla_e\phi \\
&= \int_{\partial C} \hat{\epsilon}_{cd}\left(\frac{1}{\kappa}\frac{\chi^2}{\rho^2}\right)\left[\frac{|\chi|}{\rho}\rho_b\rho^e - \left(\frac{\rho}{|\chi|} + t\cdot\rho\right)\chi_b\chi^e\right] \\
&\quad \times (\mathcal{T}_2 D\mathcal{T}_1 - \mathcal{T}_1 D\mathcal{T}_2)\nabla^b\phi\nabla_e\phi \\
&= 0, \qquad (2.29)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\partial C} \epsilon_{abcd}\xi_2^a\xi_1^b\mathbf{L} \\
&= \int_{\partial C} \hat{\epsilon}_{cd}\mathbf{L}\left[\frac{|\chi|}{\rho}\mathcal{T}_2\rho_b + \left(\frac{\rho}{|\chi|} + t\cdot\rho\right)\mathcal{R}_2\chi_b\right](\mathcal{T}_1\chi^b + \mathcal{R}_1\rho^b) \\
&= \int_{\partial C} \hat{\epsilon}_{cd}\mathbf{L}\left[\frac{|\chi|}{\rho}\mathcal{T}_2\mathcal{R}_1\rho^2 + \left(\frac{\rho}{|\chi|} + t\cdot\rho\right)\mathcal{R}_2\mathcal{T}_1\chi^2\right] \\
&= 0, \qquad (2.30)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\partial C} \epsilon_{abcd}R_e^b\left(\xi_1^a\xi_2^e - \xi_2^a\xi_1^e\right) \\
&= \int_{\partial C} \hat{\epsilon}_{cd}R_e^b\left(\frac{1}{\kappa}\frac{\chi^2}{\rho^2}\right)\left[\frac{|\chi|}{\rho}\rho_b\rho^e - \left(\frac{\rho}{|\chi|} + t\cdot\rho\right)\chi_b\chi^e\right] \\
&\quad \times (\mathcal{T}_1 D\mathcal{T}_2 - \mathcal{T}_2 D\mathcal{T}_1) = 0. \qquad (2.31)
\end{aligned}
$$

Substituting Eqs. (2.25), (2.29), (2.30), and (2.31) into Eq. (2.28) we get

$$
\delta_{\xi_2}H[\xi_1] = \int_{\partial C} \epsilon_{abcd}\left[\xi_2^a\nabla_e\left(\nabla^e\xi_1^b - \nabla^b\xi_1^e\right) - \xi_1^a\nabla_e\left(\nabla^e\xi_2^b - \nabla^b\xi_2^e\right)\right]. \qquad (2.32)
$$

The left side of Eq. (2.28) can be interpreted as the variation of the boundary term J since the "bulk" part of the generator $H[\xi_1]$ on the left side vanishes on shell. On the other hand, the change in $J[\xi_1]$ under a surface deformation generated by $J[\xi_2]$ can be precisely described by Dirac bracket $\{J[\xi_1], j[\xi_2]\}^*$ (Carlip, 1999b).

Thus we arrive at

$$\{J[\xi_1], J[\xi_2]\}^* = \int_{\partial C} \epsilon_{abcd} \left[\xi_2^a \nabla_e\left(\nabla^e \xi_1^b - \nabla^b \xi_1^e\right) - \xi_1^a \nabla_e\left(\nabla^e \xi_2^b - \nabla^b \xi_2^e\right)\right]. \tag{2.33}$$

Substituting Eqs. (2.19), (2.20), and (2.22) into (2.33) we have

$$\{J[\xi_1], J[\xi_2]\}^* = -\int_{\partial C} \hat{\epsilon}_{cd} \left[\frac{1}{\kappa}(\mathcal{T}_1 D^3 \mathcal{T}_2 - \mathcal{T}_2 D^3 \mathcal{T}_1 - 2\kappa(\mathcal{T}_1 D \mathcal{T}_2 - \mathcal{T}_2 D \mathcal{T}_1)\right]. \tag{2.34}$$

The Hamiltonian (2.5) consists of two terms, but Eqs. (2.24) and (2.25) and discussion about $\xi \cdot \mathbf{\Theta}$ (Carlip, 1999b) show that the second terms make no contribution. Thus, we have

$$J[\{\xi_1, \xi_2\}] = \int_{\partial C} \hat{\epsilon}_{cd} \left[2\kappa(\mathcal{T}_1 D \mathcal{T}_2 - \mathcal{T}_2 D \mathcal{T}_1) - \frac{1}{\kappa} D(\mathcal{T}_1 D^2 \mathcal{T}_2 - \mathcal{T}_2 D^2 \mathcal{T}_1)\right]. \tag{2.35}$$

On shell Eq. (2.6) can be expressed as

$$\{J[\xi_1], J[\xi_2]\}^* = J[\{\xi_1, \xi_2\}] + K[\xi_1, \xi_2]. \tag{2.36}$$

We know that from Eqs. (2.34) and (2.35) the central term is given by

$$K[\xi_1, \xi_2] = \int_{\partial C} \hat{\epsilon}_{cd} \frac{1}{\kappa}(D \mathcal{T}_1 D^2 \mathcal{T}_2 - D \mathcal{T}_2 D^2 \mathcal{T}_1). \tag{2.37}$$

In next section, we will study statistical entropies of some static dilaton black holes using the constraint algebra.

3. STATISTICAL ENTROPY OF STATIC DILATON BLACK HOLES

3.1. The Garfinkle–Horowitz–Strominger Dilatonic Black Hole

One of the solutions for Eqs. (2.8), (2.9), and (2.10) is the Garfinkle–Horowitz–Strominger (GHS) dilatonic black hole (Garfinkle *et al.*, 1991), which can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r(r-a)(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{3.1}$$

with

$$e^{-2\phi} = e^{-2\phi_0} - \frac{Q^2}{Mr}, \tag{3.2}$$

$$F = Q \sin\theta\, d\theta \wedge d\varphi, \tag{3.3}$$

here $a = \frac{Q^2}{2M} e^{-2\phi_0}$, and Q is the magnetic charge. If we consider functions of υ with period T, a one-parameter group of diffeomorphism satisfying Equation (2.21) can be taken as

$$\mathcal{T}_n = \frac{T}{2\pi} \exp\left[\frac{2\pi n i}{T}\upsilon\right]. \tag{3.4}$$

Substituting Eq. (3.4) into central term (2.37), and using condition (2.21) we obtain

$$K[\mathcal{T}_m, \mathcal{T}_n] = -\frac{i A_{\mathrm{H}}}{8\pi} \frac{2\pi}{\kappa T} m^3 \delta_{m+n,0}, \tag{3.5}$$

where $A_{\mathrm{H}} = \int_{\partial C} \hat{\epsilon}_{cd} = 4\pi r_+(r_+ - a)$ is the area of the event horizon. Equation (2.36) thus takes standard form of a Virasoro algebra

$$i\{J[\mathcal{T}_m], J[\mathcal{T}_n]\} = (m - n)J[\mathcal{T}_{m+n}] + \frac{c}{12} m^3 \delta_{m+n,0}, \tag{3.6}$$

with central charge

$$\frac{c}{12} = \frac{A_{\mathrm{H}}}{8\pi} \frac{2\pi}{\kappa T}. \tag{3.7}$$

The boundary term $J[\mathcal{T}_0]$ can easily be obtained by using Eqs. (2.3), (2.13), and (3.4), which is given by

$$J[\mathcal{T}_0] = \Delta = \frac{A_{\mathrm{H}}}{8\pi} \frac{\kappa T}{2\pi}. \tag{3.8}$$

Thus, from standard Cardy's formula (Carlip, 1999b)

$$\rho(\Delta) \sim \exp\left\{2\pi \sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)}\right\}, \tag{3.9}$$

we know that the number of states with a given eigenvalue Δ of $J[\mathcal{T}_0]$ grows asymptotically for large Δ as

$$\rho(\Delta) \sim \exp\left[\frac{A_{\mathrm{H}}}{4}\sqrt{2 - \left(\frac{2\pi}{\kappa T}\right)^2}\right]. \tag{3.10}$$

If and only if we take the period T as the periodicity of the Euclidean black hole, i.e.,

$$T = \frac{2\pi}{\kappa}, \tag{3.11}$$

the statistical entropy of the black hole

$$S_0 \sim \ln \rho(\Delta) = \frac{A_{\mathrm{H}}}{4} = \pi r_+(r_+ - a), \tag{3.12}$$

coincides with the standard Bekenstein–Hawking entropy.

Now let us consider the first-order quantum correction to the entropy. In order to do that, we should use following Cardy formula (see appendix for detail)

$$\rho_{cq}(\Delta) \approx \left[\frac{c_{\text{eff}}}{96\left(\Delta - \frac{c}{24}\right)^3} \right]^{1/4} \exp\left\{ 2\pi \sqrt{\frac{c_{\text{eff}}}{6}\left(\Delta - \frac{c}{24}\right)} \right\} \rho(\Delta_0), \tag{3.13}$$

where $c_{\text{eff}} = c - 24\Delta_0$. Then, from Eqs. (3.7), (3.8), and (3.11) we know that the entropy is given by

$$\begin{aligned} S &= \frac{A_{\text{H}}}{4} - \frac{3}{2}\ \ln\ \frac{A_{\text{H}}}{4} + \ln\ c + \text{const}, \\ &= \frac{A_{\text{H}}}{4} - \frac{1}{2}\ \ln\ \frac{A_{\text{H}}}{4} + \text{const}. \end{aligned} \tag{3.14}$$

The first line has two logarithmic terms and agrees with Carlip's results (1.2) (Carlip, 2000). However, after we take $T = 2\pi/\kappa$, the second line shows that the factor of the logarithmic term becomes $-\frac{1}{2}$, which is different from Kaul and Majumdar's result.

3.2. The Garfinkle–Maeda Dilaton Black Hole

The Garfinkle–Maeda (GM) dilaton black hole metric obtained from string theory (2.7) can be expressed as (Garfinkle *et al.*, 1991; Gibbons and Maeda, 1988)

$$\begin{aligned} ds^2 = &-\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{1-\alpha^2}{1+\alpha^2}} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{\frac{\alpha^2-1}{1+\alpha^2}} dr^2 \\ &+ r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} (d\theta^2 + \sin^2\theta\ d\varphi^2), \end{aligned} \tag{3.15}$$

with dilaton field

$$e^{2\Phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2\alpha}{1+\alpha 2}} e^{-2\Phi_0}, \tag{3.16}$$

and Maxwell field

$$F = \frac{Q}{r^2}\, dt \wedge\ dr, \tag{3.17}$$

where $r = r_+$ is the location of the event horizon. For $\alpha = 0$, $r = r_-$ is the location of the inner Cauchy horizon; however, for $\alpha > 0$ the surface $r = r_-$ is singular. The mass M and charge Q of the black hole are related to parameters r_+ and r_- by $2M = r_+ + (\frac{1-\alpha^2}{1+\alpha^2})r_-$, and $Q^2 = \frac{r+r-}{1+\alpha^2}\ e^{2a\Phi_0}$.

The black hole has a Killing vector $\chi^a \frac{\partial}{\partial x^a} = \partial_t$. Thus, we obtain a one-parameter group of diffeomorphism

$$\mathcal{T}_n = \frac{T}{2\pi} \exp\left[\frac{2\pi ni}{T} \upsilon\right]. \tag{3.18}$$

Using the standard Cardy formula, the analysis of the preceding subsection goes through with virtually no changes, yields a statistical entropy

$$S_0 = \frac{A_{\rm H}}{4} = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{2\alpha^2/(1+\alpha^2)}. \tag{3.19}$$

The result is equal to the Bekenstein–Hawking entropy.

However, if we consider first-order quantum correction to the entropy by the new Cardy formula (3.13), we get

$$S = \frac{A_{\rm H}}{4} - \frac{1}{2}\ln\frac{A_{\rm H}}{4} + {\rm const.} \tag{3.20}$$

4. SUMMARY AND DISCUSSION

With Carlip's boundary conditions, a standard Virasoro subalgebra with corresponding central charge for static dilaton black hole is constructed at a Killing horizon. As examples, the Garfinkle–Horowitz–Strominger dilaton black hole and the Gibbons–Maeda dilaton black hole are considered. Only if we take T as the periodicity of the Euclidean black hole, $T = 2\pi/\kappa$, the statistical entropies of the black holes yielded by standard Cardy formula agree with their Bekenstein–Hawking entropies. It is easy to show that the result can be used for other static dilaton black holes. Therefore, Carlip's conclusion—the asymptotic behavior of the density of states may be determined by the algebra of diffeomorphism at horizon—is valid for static dilaton black holes obtained from the low-energy effective field theory that in turn is obtained from heterotic string theory.

If we consider first-order quantum correction the entropy contains extra logarithmic terms which agree with Carlip's results (Carlip, 2000). However, we know that in order to get the Bekenstein–Hawking entropy we have to take $T = 2\pi/\kappa$. That is to say, we can not set central charge c to be a universal constant, independent of area of the event horizon, by adjusting periodicity T as Carlip did (Carlip, 2000). Therefore, the factor of the logarithmic term is $-\frac{1}{2}$, which is different from Kaul and Majumdar's factor of $-\frac{3}{2}$.

It is well-known that leading correction to the entropy of the black hole is described by a logarithmic term (Carlip, 2000; Cognola, 1998; Frolov and Fursaev, 1998; Fursaev and Solodukhin, 1996; Jing and Yan, 1999, 2000; Kaul and Majumdar, 2000; Mann and Solodukhin, 1996). We do not think that a real physical result is related to calculating approach. Therefore, the reason that different methods lead to different corrections to the Bekenstein–Hawking entropy should be sought deeply.

APPENDIX: LOGARITHMIC CORRECTIONS TO THE CARDY FORMULA

Carlip (2000) has shown that the number of states is

$$\rho(\Delta) = \int d\tau\, e^{-2\pi i \Delta \tau}\, e^{-2\pi i \Delta_0 \frac{1}{\tau}}\, e^{\frac{2\pi i c}{24}\tau}\, e^{\frac{2\pi i c}{24}\frac{1}{\tau}}\, \tilde{Z}(-1/\tau), \tag{A1}$$

where $\tilde{Z}(-1/\tau)$ approaches to a constant, $\rho(\Delta_0)$, for large τ. So the integral (A1) can be evaluated by steepest descent provided that the imaginary part of τ is large at the saddle point. The integral takes the form

$$I[a, b] = \int d\tau\, e^{2\pi i a\tau + \frac{2\pi i b}{\tau}}\, f(\tau). \tag{A2}$$

The argument of the exponent is extremal at $\tau_0 = \sqrt{\frac{b}{a}}$, and expanding around τ_0, one has (Carlip, 2000)

$$I[a, b] \approx \int d\tau\, e^{4\pi i a\sqrt{ab} + \frac{2\pi i b}{\tau_0^3}(\tau-\tau_0)^2}\, f(\tau_0) = \left(-\frac{b}{4a^3}\right)^{1/4} e^{4\pi i\sqrt{ab}}. \tag{A3}$$

Comparing Eqs. (A1) with (A2) we know

$$a = \frac{c}{24} - \Delta, \qquad b = \frac{c}{24} - \Delta_0. \tag{A4}$$

Therefore, if we let $C_{\text{eff}} = c - 24\Delta_0$, the Cardy formula including logarithmic corrections can be expressed as

$$\rho_{cq}(\Delta) \approx \left[\frac{C_{\text{eff}}}{96\left(\Delta - \frac{c}{24}\right)^3}\right]^{1/4} \exp\left\{2\pi\sqrt{\frac{C_{\text{eff}}}{6}\left(\Delta - \frac{c}{24}\right)}\right\} \rho(\Delta_0), \tag{A5}$$

which is equal to Carlip's result (C.3) in Appendix C (Carlip, 1999b), if we ignore the lowest order correction.

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